

# Geometrizing the Quantum - A Toy Model

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It is shown that the equations of relativistic Bohmian mechanics for multiple bosonic particles have a dual description in terms of a classical theory of conformally “curved” space-time. This shows that it is possible to formulate quantum mechanics as a purely classical geometrical theory. The results are further generalized to interactions with an external electromagnetic field.

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## A. Introduction

In standard quantum mechanics observables and the corresponding uncertainty are promoted to a fundamental principle. But it was shown by David Bohm that this does not necessarily have to be the case [1]. In the de Broglie-Bohm (dBB) interpretation it is explained that the uncertainty “principle” and the description by means of operators can be understood in terms of uncontrollable initial conditions and non-local interactions between an additional field (the “pilot wave”) and the measuring apparatus. This theory was further generalized to relativistic quantum mechanics and quantum field theory with bosonic and fermionic fields [2–5]. It is well known that (due to its non-locality) the dBB theory is not in contradiction to the Bell inequalities [6]. Due to its contextuality, the dBB theory is also not affected by the Kochen-Specker theorem [7].

One mayor drawback of the dBB theory is that the pilot wave and the corresponding “quantum potential” have to be imposed by hand without further justification. In a previous work [17] it was shown that the relativistic dBB theory for a single particle is dual to a scalar theory of curved spacetime. In this dual theory the ominous “pilot wave” can be readily interpreted as a well known physical quantity, namely a space-time dependent conformal factor of the metric. This work on the single particle has many features in common with other publications on the subject [9–14]. However, having a duality for the single particle case is not enough, because the dBB interpretation only is a consistent quantum theory when it also has the many particle case. The many particle theory is for example crucial for understanding the quantum uncertainty and for evading the non existence theorems [6, 7]. Therefore, we will generalize the previous results and present a dual for the relativistic many particle dBB theory.

## B. Relativistic dBB for many particles

In this section we shortly list the ingredients for the interpretation of the many particle quantum Klein-Gordon equation in terms of Bohmian trajectories. For a detailed description of subsequent topics in the dBB theory like particle creation, the theory of quantum measurement, many particle states, and quantum field theory the reader is referred to [5]. Let  $|0\rangle$  be the vacuum and  $|n\rangle$  be an arbitrary  $n$ -particle state. The corresponding  $n$ -particle wave function is [3]  $\psi(x_1; \dots; x_n) = \frac{\mathcal{P}_s}{\sqrt{n!}} \langle 0 | \hat{\Phi}(x_1) \dots \hat{\Phi}(x_n) | n \rangle$ , where the  $\hat{\Phi}(x_j)$  are scalar Klein-Gordon field operators and the symbol  $\mathcal{P}_s$  denotes symmetrization over all positions  $x_j$ . For free fields, the wave function satisfies the equation

$$\left( \sum_j^n \partial_j^m \partial_{jm} + n \frac{M^2}{\hbar^2} \right) \psi(x_1; \dots; x_n) = 0 \quad . \quad (1)$$

The mass of a single particle is given by  $M$ . The index  $j$  indicates which one of the  $n$  particles is affected and the index  $m$  is the typical space-time index in four flat dimensions. The wave function allows further for the construction of a conserved current  $\sum_j^n \partial_j^m (i\psi^* \partial_{jm} \psi - i\psi \partial_{jm} \psi^*) = 0$ . This is still standard quantum mechanics, in order to arrive at the dBB interpretation one rewrites the wave function  $\psi = P e^{iS/\hbar}$  by introducing a real amplitude  $P(x_1; \dots; x_n)$

and a real phase  $S(x_1; \dots; x_n)$ . Doing this, the complex equation (1) splits up into two real equations

$$2MQ \equiv \sum_j^n (\partial_j^m S)(\partial_{jm} S) - nM^2, \quad \text{with } Q = \sum_j^n \frac{\hbar^2}{2M} \frac{\partial_j^m \partial_{jm} P}{P} \quad (2)$$

$$0 \equiv \sum_j^n \partial_{jm} (P^2 (\partial_j^m S)) \quad , \quad (3)$$

where  $Q(x_1; \dots; x_n)$  is the quantum potential, and  $P(x_1; \dots; x_n)$  is the pilot wave. The first equation can be interpreted as a classical Hamilton-Jacobi equation with the additional potential  $Q$  and the second equation takes the form of a conserved current. This is the only way that the  $\hbar$  enters into the dBB theory. In the dBB interpretation one postulates the existence of particle trajectories  $x_j^m(s)$  whose momentum  $p_j^m$  satisfies the relation

$$p_j^m = M \frac{dx_j^m}{ds} \equiv -\partial_j^m S \quad . \quad (4)$$

Now one can derive this expression with respect to  $ds$  and use the identity

$$\frac{d}{ds} = \sum_j^n \frac{dx_j^m}{ds} \partial_{jm} \quad . \quad (5)$$

This gives the equation of motion for all  $n$  relativistic particles in the dBB interpretation

$$\frac{d^2 x_j^m}{ds^2} = \sum_i^n \frac{(\partial_i^l S)(\partial_j^m \partial_{il} S)}{M^2} \quad . \quad (6)$$

By using equation (2) this can be further simplified to

$$M \frac{d^2 x_j^m}{ds^2} = \partial_j^m Q \quad . \quad (7)$$

The infinitesimal parameter  $s$  is not necessarily time, because every single particle carries its own reference frame. For convenience one might try to choose  $s$  as the eigen-time of the particle which is finally subject to a measurement. The non-local nature of the dBB theory becomes obvious in the above equations of motion. The trajectory of the particle  $j$  is determined from the potential  $Q(x_1, \dots, x_n)$ , which depends on the positions of all other particles of the system.

The equations (3-6) are the building blocks of the many particle dBB theory. The functions  $P$ ,  $S$ , and  $Q$  that appear in those equations depend on the  $4 \times n$  coordinates  $x_j^m$ . It is therefore possible to introduce a single  $4n$  dimensional coordinate  $x^L = (x_1^0, x_1^1, x_1^2, x_1^3; \dots; x_n^0, x_n^1, x_n^2, x_n^3)$ , which has a capital Latin index and contains the space-time positions of all  $n$  particles. One further observes that in all equations every summation over a particle index  $j$  is accompanied by a summation over the space-time index  $m$  of the corresponding particle. This allows to replace  $\partial_{jm} \rightarrow \partial_L$  and  $\partial_j^m \rightarrow \partial^L$ . Thus, one can rewrite the equations for the many particle case (2-6) as

$$2MQ \equiv (\partial^L S)(\partial_L S) - nM^2 \quad \text{with } Q \equiv \frac{\hbar^2}{2M} \frac{\partial^L \partial_L P}{P}, \quad (8)$$

$$0 \equiv \partial_L (P^2 (\partial^L S)) \quad , \quad (9)$$

$$p^L \equiv M \frac{dx^L}{ds} \equiv -\partial^L S \quad , \quad (10)$$

$$\frac{d^2 x^L}{ds^2} = \frac{(\partial^N S)(\partial^L \partial_N S)}{M^2} \quad \text{with } \frac{d}{ds} \equiv \frac{dx^L}{ds} \partial_L \quad . \quad (11)$$

### C. A $4 \times n$ dimensional toy model

We will now show that the equations of the many particle dBB theory (8-11) have a dual description in a scalar theory of curved space-time in  $4 \times n$  dimensions. As generalization of the previous single particle approach [17] we will define a setup where the momentum of every particle is defined in the particles own four dimensional space-time. Such

a  $n$ -particle theory is therefore defined in a  $4 \times n$  dimensional space-time. Following the flat notation the coordinates in curved space-time will be denoted as  $\hat{x}^\Lambda = (\hat{x}_1^0, \hat{x}_1^1, \hat{x}_1^2, \hat{x}_1^3; \dots; \hat{x}_n^0, \hat{x}_n^1, \hat{x}_n^2, \hat{x}_n^3)$ .

The theory can be formulated by starting from a  $n$ -particle equation of motion

$$\mathcal{P}_s[\hat{R}] = \mathcal{P}_s[\hat{T}] \quad . \quad (12)$$

Here,  $\hat{R}$  is the Ricci scalar,  $\hat{\mathcal{L}}_M$  is the stress energy tensor of matter, and  $\kappa$  is the coupling constant of this theory.  $\mathcal{P}_s$  is a symmetrization operator between different particles  $x_i^\lambda$  and  $x_j^\lambda$ . This means that the fields that are a solution of the problem have to be symmetric under exchange of two particle coordinates  $x_i^\lambda$  and  $x_j^\lambda$ . It also means that all the different particles agree on their definition of what is the  $x_j^\mu$  direction. Therefore, if one wants to perform a coordinate transformation in a single four dimensional subspace, one has to perform the same transformation in all other four dimensional subspaces. In order to describe the local conformal part of this theory separately one splits the metric  $\hat{g}$  up into a conformal function  $\phi(x)$  and a non-conformal part  $g$

$$\hat{g}_{\Lambda\Gamma} = \phi^{\frac{2}{2n-1}} g_{LG} \quad . \quad (13)$$

The index notation for tensors used here is explained in table (I) and it allows to write the equations in  $D = 4 \times n$  dimensions either with one index (capital letters), or with two indices (lower case letters). A further distinction is made between indices that are shifted by the metric  $\hat{g}$  (Greek) and indices that are shifted by the metric  $g$  (Latin). In

Index notation	Shifted by $\hat{g}$	Shifted by $g$
Single index	$\hat{X}_\Lambda; \dots   \{ \Lambda : 1 \dots 4n \}$	$X_L; \dots   \{ L : 1 \dots 4n \}$
Double index	$\hat{X}_{i\lambda}; \dots   \{ i : 1 \dots n \}$	$X_{il}; \dots   \{ i : 1 \dots n \}$

TABLE I: Index convention

this section we will only use the single index notation, but all results can be immediately translated into the double index notation used in the previous section. The inverse of the metric (13) is  $\hat{g}^{\Lambda\Gamma} = \phi^{-\frac{2}{2n-1}} g^{LG}$ . Indices with a lower Greek and a lower Roman index can be identified  $\hat{\partial}_\Lambda \equiv \partial_L$ . From this follows for example that the adjoint derivatives are not identical, in both notations

$$\hat{\partial}^\Lambda = \hat{g}^{\Lambda\Sigma} \hat{\partial}_\Sigma = \phi^{-\frac{2}{2n-1}} g^{LS} \partial_S = \phi^{-\frac{2}{2n-1}} \partial^L \quad . \quad (14)$$

The geometrical dual to the first dBB equation: The definition of the metric (13) allows to reformulate the model in terms of the separate functions  $\phi$  and  $g_{LD}$  [15]. Keeping in mind the symmetrization condition the equation (12) reads  $\frac{2(4n-1)}{1-2n} (\partial^L \partial_L \phi) = \phi(-R + \kappa \mathcal{T}_M)$ . We are primarily interested in studying a flat Minkowski background space-time  $g_{LD} = \eta_{LD}$ . This can be achieved by adding an additional condition term to (12) that demands a vanishing Weyl curvature. Such a condition also appears in the scalar theories of curved space-time suggested by Gunnar Nordström [16]. Like in standard general relativity one further imposes that the metric has a vanishing covariant derivative. From those conditions it follows that the metric  $g_{LG}$  has only  $\pm 1$  on the diagonal, while all other entries vanish  $g_{LG} = \eta_{LG}$ . Thus,  $R = 0$ , which simplifies the above equation to

$$\frac{2(4n-1)}{1-2n} (\partial^L \partial_L \phi) = \kappa \phi \mathcal{T}_M \quad . \quad (15)$$

An Extension of the Hamilton Jacobi stress energy tensor  $\hat{\mathcal{T}}_M$  can be defined by subtracting a mass term  $\hat{M}^2$  for every particle on finds

$$\begin{aligned} \hat{\mathcal{T}}_M &= \hat{p}^\Lambda \hat{p}_\Lambda - n \hat{M}_G^2 \\ &= \phi^{\frac{-2}{2n-1}} ((\partial^L S_H)(\partial_L S_H) - n M_G^2) \quad . \end{aligned} \quad (16)$$

The Hamilton principle function  $S_H$  defines the local momentum  $\hat{p}^\Lambda = \hat{M}_G d\hat{x}^\Lambda / d\hat{s} = -\hat{\partial}^\Lambda S_H$ . Plugging this into equation (15) gives

$$\frac{2(4n-1)}{\kappa(1-2n)} \frac{\partial^L \partial_L \phi}{\phi} = (\partial^L S_H)(\partial_L S_H) - n M_G^2 \quad (17)$$

Now one can see that this is exactly the first dBB equation (8) if one identifies

$$\begin{aligned} \phi(x) &= P(x) \quad , \quad S_H(x) = S(x) \quad , \\ \kappa &= \frac{2(4n-1)}{1-2n}/\hbar^2 \quad , \quad M^2 = M_G^2 \quad . \end{aligned} \quad (18)$$

Note that the matching conditions demand like in the single particle case [17] a negative coupling  $\kappa$ .

The geometrical dual to the third dBB equation: According to the Hamilton-Jacobi formalism the derivatives of the Hamilton principle function ( $S_H$ ) define the momenta

$$\hat{p}_\Lambda \equiv -(\partial_\Lambda S_H) \quad . \quad (19)$$

Therefore, with the prescription (14) and the matching condition (18) one sees immediately that the third Bohmian equation (10) is fulfilled.

The geometrical dual to the second dBB equation: In order to find the dual to the second Bohmian equation one has to exploit that the stress-energy tensor (16) is covariantly conserved  $\hat{\nabla}_\Lambda \hat{T}^{\Lambda\Delta} = 0$ . This is true if the following relations are fulfilled

$$\hat{\nabla}_\Lambda (\hat{\partial}^\Lambda S_H) = 0, \quad (\hat{\partial}^\Lambda S_H) \hat{\nabla}^\Delta (\hat{\partial}_\Lambda S_H) = 0, \quad (\hat{\partial}^\Lambda S_H) \hat{\nabla}_\Lambda (\hat{\partial}^\Delta S_H) = 0. \quad (20)$$

Now one needs to know the Levi Civita connection

$$\Gamma_{\Lambda\Delta}^\Sigma = \frac{1}{2} \phi^{-\frac{2}{2n-1}} \left[ (\partial_L \phi^{\frac{2}{2n-1}}) \delta_D^S + (\partial_D \phi^{\frac{2}{2n-1}}) \delta_L^S - (\partial^S \phi^{\frac{2}{2n-1}}) \eta_{LD} \right] \quad . \quad (21)$$

With this, the condition (20) reads

$$\hat{\nabla}_\Lambda (\hat{\partial}^\Lambda S_H) = \phi^{-\frac{4n}{2n-1}} \partial_L [\phi^2 (\partial^L S_H)] = 0 \quad . \quad (22)$$

With the matching conditions (18), the above equation is identical to the second Bohmian equation (9).

The geometrical dual to the dBB equation of motion: The total derivative (5) is generalized to  $\frac{d}{ds} = \frac{d\hat{x}^\Lambda}{ds} \hat{\partial}_\Lambda = \phi^{2/(1-2n)} \frac{dx^L}{ds} \partial_L = \phi^{2/(1-2n)} \frac{d}{ds}$ . Applying this to the momentum  $\hat{M}_G(d\hat{x}^\Lambda/ds) = (\hat{\partial}^\Lambda S_H)$  gives the equation of motion  $\hat{M} \frac{d^2 \hat{x}^\Lambda}{ds^2} = \frac{1}{M} (\hat{\partial}^\Delta S_H) \hat{\partial}_\Delta (\hat{\partial}^\Lambda S_H)$ , or equivalently the equation of motion in the Minkowski coordinates  $x^L$

$$\frac{d^2 x^L}{ds^2} = \frac{(\partial^N S_H) (\partial^L \partial_N S_H)}{M^2} \quad . \quad (23)$$

Using (18) one sees that the equation of motion (23) is dual to the equation of motion of relativistic Bohmian mechanics (11). This is however almost a triviality because the equations (11) and (23) are all derived from the same mathematical prescription (5). But in curved space-time there is an other equation of motion in addition to the equation of motion (23), the geodesic equation which can be shown to lead to the same result.

### 1. Interaction with an external field

Now, the results of the previous sections will be generalized to interactions with an external electromagnetic field.

An external field in the dBB theory: Coupling  $n$  bosonic particles (1) with charge  $e$  to an external electromagnetic field  $A_m$  is achieved by replacing the partial derivative with a gauge covariant derivative  $\partial_m \rightarrow \partial_m + ieA_m/\hbar$  in the Klein-Gordon Lagrangian. The resulting equation of motion is

$$\sum_j^n \left[ \left( \partial_j^m \partial_{jm} + \frac{M}{\hbar^2} - \frac{e^2 A_j^2}{\hbar^2} \right) \psi + \frac{i \partial_j^m (\psi^2 A_{jm})}{\hbar} \right] = 0 \quad , \quad (24)$$

where  $A_j^m$  is the electromagnetic potential  $A^m$  evaluated at the position of particle  $j$ . By again rewriting the  $n$ -particle wave function  $\psi = P \exp(iS)/\hbar$  the two equations (2, 3) generalize to

$$2MQ \equiv \sum_j^n (\partial_j^m S + eA_j^m) (\partial_{jm} S + eA_{jm}) - nM^2, \quad (25)$$

$$0 \equiv \sum_j^n \partial_{jm} (P^2 (\partial_j^m S + eA_j^m)) \quad . \quad (26)$$

In the presence of an external force, the dBB definition for the particles momentum (4) now contains the canonical momentum  $\pi_j^m$  instead of the normal momentum  $p_j^m$

$$\pi_j^m = M \frac{dx_j^m}{d\tau} \equiv -(\partial_j^m S + eA_j^m) \quad . \quad (27)$$

Thus, using (5, 25, 27, and the relation  $\partial_{kn}A_j^m = \delta_{kj}\partial_nA_j^m$ ) one finds the equation of motion for all  $n$  particles in an external field  $A^m$

$$M \frac{d^2x_j^m}{ds^2} = \partial_j^m Q + e\pi_{jn}F^{mn} \quad . \quad (28)$$

Here, the field strength tensor is  $F_{jk}^{mn} = \partial_j^m A_k^n - \partial_k^m A_j^n = \delta_{jk}F^{mn}$  with  $F^{mn} = \partial^m A^n - \partial^n A^m$ . On the RHS of equation (28) appear two terms. The first term is the quantum potential also present in equation (7) and the second term is the Lorentz force which is familiar from classical electrodynamics. Now one can undo the formal rewriting of coordinates and the interacting equations (25-28) read

$$2MQ \equiv (\partial^L S + eA^L)(\partial_L S + eA_L) - nM^2 \quad (29)$$

$$0 \equiv \partial_L (P^2(\partial^L S + eA^L)) \quad , \quad (30)$$

$$p^L \equiv M \frac{dx^L}{ds} \equiv -(\partial^L S + eA^L) \quad , \quad (31)$$

$$M \frac{d^2x^L}{ds^2} = \partial^L Q + e\pi_K F^{LK} \quad . \quad (32)$$

An external field in the  $4 \times n$  dimensional theory of curved space-time: In the interacting case, the classical  $4 \times n$  dimensional theory of curved space-time is analogous to the discussion in section C. The only difference appears in the definition of the canonical momentum  $\hat{\pi}^\Lambda = \hat{M}_G \frac{d\hat{x}^\Lambda}{ds} = -(\hat{\partial}^\Lambda S_H + e\hat{A}^\Lambda)$ , instead of the free momentum  $\hat{p}^\Lambda$ . With this replacement the equations (17,22,and 19) transform to

$$\frac{2(4n-1)}{\kappa(1-2n)} \frac{\partial^L \partial_L \phi}{\phi} \equiv (\partial^L S_H + eA^L)(\partial_L S_H + eA_L) - nM_G^2 \quad , \quad (33)$$

$$0 \equiv \phi^{\frac{4n}{1-2n}} \partial_L (P^2(\partial^L S_H + eA^L)) \quad . \quad (34)$$

One immediately sees that with the identifications (18) the above equations are dual to the equations (29-31) In order to check the equation of motion (32) we use the total derivative and find  $\frac{d^2\hat{x}^\Lambda}{ds^2} = \phi^{\frac{4}{1-2n}} \frac{d^2x^L}{ds^2} + \hat{\pi}^L \cdot (\dots)$ . Using this and  $\hat{\pi}^\Lambda \hat{\nabla}_\Lambda \hat{\pi}^\Delta = 0$  one can verify that the geodesic equation is consistent with (32).

## 2. Discussions

In this section some interpretational issues are addressed. This aims to develop a physical understanding of the presented duality.

**Locality:** Quantum mechanics in the dBB interpretation is a theory which allows to talk about particle position at the cost of non-local interactions. The non-locality becomes obvious by looking at the equation of motion for a free particle in the dBB theory (7). The movement of one particle is governed by the quantum potential  $Q$ , which simultaneously depends on the positions of all the other particles of the system. The next question is: “How can a non-local theory have a dual local theory?” In the presented theory, every single particle is living in its own four dimensional space-time. Therefore, the positions of  $n$  different particles correspond to one single point in the  $4 \times n$  dimensional space-time. This higher dimensional construction helps around the non-locality argument, but it has the price that the theory has to be formulated in the  $4 \times n$  configuration space.

**Time:** One notes the appearance of  $n$  time coordinates in this formulation. However, this possible conceptual problem can be evaded since the actual quantum observables are restricted by equal time commutation relations. They form a subset of the solutions allowed here. One might therefore only consider solutions where all  $x_i^0$  are equal. In other approaches that differ from this one it is put forward that the nature of quantum mechanics could be a consequence of an additional time dimension [17, 18].

**Gravity:** “How is this geometrical theory related to THE geometrical theory - general relativity?” It is tempting to speculate about a generalization of the given toy model to tensor equations. However, writing the analogous higher

dimensional tensor equations leads to a large set of strongly coupled differential equations. It will have to be explored whether in some limit, classical general relativity and the dBB theory are part of the space of solutions.

The matching conditions: The matching conditions (18) are not unique. They were chosen in order to have a direct connection between three functions. This has the consequence that if one gives  $\hbar$  a fixed numerical value, then the coupling of the geometrical theory  $\kappa$  runs from  $-6/\hbar^2$  to  $-4/\hbar^2$ , depending on the number of particles ( $n = 1, \dots, \infty$ ). Reversely demanding a fixed geometrical coupling would result in a running Planck constant  $\hbar$ . One might see the scale dependence of the coupling  $\kappa$  as a feature of the toy model which it shares with effective quantum field theories and even some approaches to gravity [19].

#### D. Summary

It was shown that the equations of the free relativistic dBB theory for many particles (8-11) have a dual description in a geometrical theory in  $4 \times n$  dimensions. For the translation between the two theories a single set of matching conditions was defined (18). The result was then generalized to interactions with an external electromagnetic field. The question whether such dualities can also be found for, self-interacting theories, fermions, or quantum field theory will be subject of future studies.

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